Lesson 11. Formulating Dynamic Programming Recursions

0 Warm up

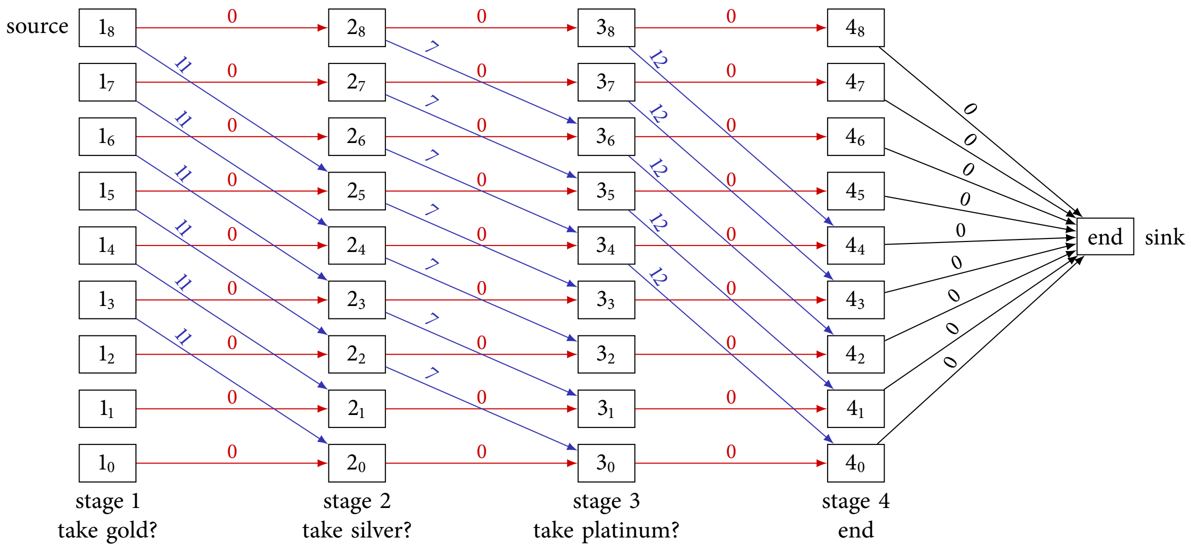
Consider the knapsack problem we studied in Lesson 5:

Example 1. You are a thief deciding which precious metals to steal from a vault:

	Metal	Weight (kg)	Value
1	Gold	3	11
2	Silver	2	7
3	Platinum	4	12

You have a knapsack that can hold at most 8kg. If you decide to take a particular metal, you must take all of it. Which items should you take to maximize the value of your theft?

- We formulated the following DP for this problem by giving the following longest path representation:



- Let $f_t(n)$ = length of a ~~shortest~~ longest path from node t_n to the end node

- In the context of the knapsack problem:

$f_1(8)$ = maximum value w/ 8 kg capacity and metals 1, 2, 3 available
= optimal value of knapsack problem w/ 8 kg knapsack and metals 1, 2, 3

$f_2(5)$ = maximum value w/ 5 kg capacity and metals 2, 3 available
= optimal value of knapsack problem w/ 5 kg knapsack and metals 2, 3.

$f_3(3)$ = maximum value w/ 3 kg capacity and metal 3 available
= optimal value of knapsack problem w/ 3 kg knapsack and metal 3

- In other words, these are optimal values of subproblems of the knapsack problem

1 Formulating DP recursions

- Last lesson: recursions for shortest path problems
- Dynamic programs are not usually given as shortest/longest path problems
    - However, it is usually easier to think about DPs this way
- Instead, the standard way to describe a dynamic program is a recursion that defines the optimal value of one subproblem in terms of the optimal values of other subproblems
- Let's formulate the knapsack problem in Example 1 as a DP, but now by giving its recursive representation
- Let

$$w_t = \text{weight of metal } t \qquad v_t = \text{value of metal } t \qquad \text{for } t = 1, 2, 3$$

- Stages:

$$\text{stage } t \longleftrightarrow \begin{cases} \text{consider taking metal } t & t = 1, 2, 3 \\ \text{end of decision-making process} & t = 4 \end{cases}$$

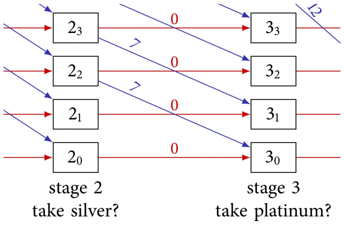
- States: state n $\longleftrightarrow$ n kg remaining in knapsack $\quad n = 0, 1, \dots, 8$

- Allowable decisions x_t at stage t and state n: $\longleftrightarrow$ edges outgoing from node t_n

x_t must satisfy

$$x_t \in \{\underset{\substack{\uparrow \\ \text{don't take} \\ \text{metal } t}}{0}, \underset{\substack{\uparrow \\ \text{take metal } t}}{1}\}$$

$$w_t x_t \leq n \leftarrow \text{we can take metal } t \text{ only if we have enough capacity.}$$



for $t = 1, 2, 3$
$n = 0, 1, \dots, 8$

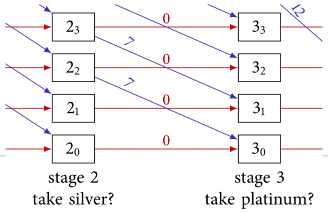
- Contribution of decision x_t at stage t and state n: $\longleftrightarrow$ edge lengths

$$v_t x_t = \begin{cases} v_t & \text{if } x_t = 1 \text{ (take metal } t) \\ 0 & \text{o/w} \end{cases} \qquad \text{for } t = 1, 2, 3 \quad n = 0, 1, \dots, 8$$

- Value-to go function $f_t(n)$ at stage t and state n:

$f_t(n)$ = maximum value w/ n kg knapsack and metals $t, t+1, \dots, 3$ remaining. $\quad$ for $t = 1, 2, 3, 4$ $\quad n = 0, 1, \dots, 8$

- Boundary conditions: $f_4(n) = 0 \quad$ for $n = 0, 1, \dots, 8$



$$f_2(3) = \max\{0 + f_3(3), 7 + f_3(1)\}$$

- Recursion:

$$f_t(n) = \max_{\substack{x_t \in \{0,1\} \\ w_t x_t \le n}} \left\{ v_t x_t + f_{t+1}(n - w_t x_t) \right\}$$

x_t allowable

for $t = 1, 2, 3$
$n = 0, 1, \ldots, 8$

- Desired value-to-go function value: $f_1(8)$

- In general, to formulate a DP by giving its recursive representation:

Dynamic program – recursive representation

- **Stages** $t = 1, 2, \ldots, T$ and **states** $n = 0, 1, 2, \ldots, N$
- Allowable **decisions** x_t at stage t and state n $\quad (t = 1, \ldots, T-1;\ n = 0, 1, \ldots, N)$
- **Contribution** of decision x_t at stage t and state n $\quad (t = 1, \ldots, T;\ n = 0, 1, \ldots, N)$
- **Value-to-go** function $f_t(n)$ at stage t and state n $\quad (t = 1, \ldots, T;\ n = 0, 1, \ldots, N)$
- **Boundary conditions** on $f_T(n)$ at state n $\quad (n = 0, 1, \ldots, N)$
- **Recursion** on $f_t(n)$ at stage t and state n $\quad (t = 1, \ldots, T-1;\ n = 0, 1, \ldots, N)$

$$f_t(n) = \min_{x_t \text{ allowable}} \text{or max} \left\{ \begin{pmatrix} \text{contribution of} \\ \text{decision } x_t \end{pmatrix} + f_{t+1}\begin{pmatrix} \text{new state} \\ \text{resulting} \\ \text{from } x_t \end{pmatrix} \right\}$$

- **Desired value-to-go function value**

- How does the recursive representation relate to the shortest/longest path representation?

Shortest/longest path		Recursive
node t_n	↔	state n at stage t
edge $(t_n, (t+1)_m)$	↔	allowable decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of edge $(t_n, (t+1)_m)$	↔	contribution of decision x_t in state n at stage t that results in being in state m at stage $t+1$
length of shortest/longest path from node t_n to end node	↔	value-to-go function $f_t(n)$
length of edges (T_n, end)	↔	boundary conditions $f_T(n)$
shortest or longest path	↔	recursion is min or max: $f_t(n) = \min_{x_t \text{ allowable}} \text{or max} \left\{ \begin{pmatrix} \text{contribution of} \\ \text{decision } x_t \end{pmatrix} + f_{t+1}\begin{pmatrix} \text{new state} \\ \text{resulting} \\ \text{from } x_t \end{pmatrix} \right\}$
source node 1_n	↔	desired value-to-go function value $f_1(n)$

2 Solving DP recursions

- To improve our understanding of how this recursive representation works, let's solve the DP we just wrote for the knapsack problem
- We solve the DP backwards:
  - start with the boundary conditions in stage T
  - compute values of the value-to-go function $f_t(n)$ in stages $T-1, T-2, \ldots, 3, 2$
  - …until we reach the desired value-to-go function value
- Stage 4 computations – boundary conditions:

$$f_4(n) = 0 \qquad \text{for} \quad n = 0, 1, \ldots, 8$$

- Stage 3 computations:

$$f_3(8) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 8}} \left\{ 12x_3 + f_4(8-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(8)},\ \overset{12}{12(1) + f_4(8-4)} \right\} = 12$$

$$f_3(7) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 7}} \left\{ 12x_3 + f_4(7-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(7)},\ \overset{12}{12(1) + f_4(7-4)} \right\} = 12$$

$$f_3(6) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 6}} \left\{ 12x_3 + f_4(6-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(6)},\ \overset{12}{12(1) + f_4(6-4)} \right\} = 12$$

$$f_3(5) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 5}} \left\{ 12x_3 + f_4(5-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(5)},\ \underset{x_3 = 1}{\overset{12}{12(1) + f_4(5-4)}} \right\} = 12$$

$$f_3(4) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 4}} \left\{ 12x_3 + f_4(4-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(4)},\ \overset{12}{12(1) + f_4(4-4)} \right\} = 12$$

$$f_3(3) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 3}} \left\{ 12x_3 + f_4(3-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(3)} \right\} = 0$$

$$f_3(2) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 2}} \left\{ 12x_3 + f_4(2-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(2)} \right\} = 0$$

$$f_3(1) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 1}} \left\{ 12x_3 + f_4(1-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(1)} \right\} = 0$$

$$f_3(0) = \max_{\substack{x_3 \in \{0,1\} \\ 4x_3 \le 0}} \left\{ 12x_3 + f_4(0-4x_3) \right\} = \max\left\{ 0 + \overset{0}{f_4(0)} \right\} = 0$$

- Stage 2 computations: $f_t(n) = \max_{\substack{x_t \in \{0,1\} \\ w_t x_t \le n}} \left\{ v_t x_t + f_{t+1}(n - w_t x_t) \right\}$ for $t = 1, 2, 3$, $n = 0, 1, \dots, 8$

$$f_2(8) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 8}} \left\{ 7x_2 + f_3(8 - 2x_2) \right\} = \max\left\{ 0 + \overset{12}{f_3(8)},\ 7 + \overset{12}{f_3(6)} \right\} = 19$$

$$f_2(7) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 7}} \left\{ 7x_2 + f_3(7 - 2x_2) \right\} = \max\left\{ 0 + \overset{12}{f_3(7)},\ 7 + \overset{12}{f_3(5)} \right\} = 19$$

$$f_2(6) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 6}} \left\{ 7x_2 + f_3(6 - 2x_2) \right\} = \max\left\{ 0 + \overset{12}{f_3(6)},\ 7 + \overset{12}{f_3(4)} \right\} = 19$$

$$f_2(5) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 5}} \left\{ 7x_2 + f_3(5 - 2x_2) \right\} = \max\left\{ 0 + \underset{x_2 = 0}{\overset{12}{f_3(5)}},\ 7 + \overset{0}{f_2(3)} \right\} = 12$$

$$f_2(4) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 4}} \left\{ 7x_2 + f_3(4 - 2x_2) \right\} = \max\left\{ 0 + \overset{12}{f_3(4)},\ 7 + \overset{0}{f_2(2)} \right\} = 12$$

$$f_2(3) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 3}} \left\{ 7x_2 + f_3(3 - 2x_2) \right\} = \max\left\{ 0 + \overset{0}{f_3(3)},\ 7 + \overset{0}{f_3(1)} \right\} = 7$$

$$f_2(2) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 2}} \left\{ 7x_2 + f_3(2 - 2x_2) \right\} = \max\left\{ 0 + \overset{0}{f_3(2)},\ 7 + \overset{0}{f_3(0)} \right\} = 7$$

$$f_2(1) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 1}} \left\{ 7x_2 + f_3(1 - 2x_2) \right\} = \max\left\{ 0 + \overset{0}{f_3(1)} \right\} = 0$$

$$f_2(0) = \max_{\substack{x_2 \in \{0,1\} \\ 2x_2 \le 0}} \left\{ 7x_2 + f_3(0 - 2x_2) \right\} = \max\left\{ 0 + \overset{0}{f_3(0)} \right\} = 0$$

- Stage 1 computations – desired value-to-go function:

$$f_1(8) = \max_{\substack{x_1 \in \{0,1\} \\ 3x_1 \le 8}} \left\{ 11x_1 + f_2(8 - 3x_1) \right\} = \max\left\{ 0 + \overset{19}{f_2(8)},\ \underset{x_1 = 1}{11 + \overset{12}{f_2(5)}} \right\} = 23$$

- Maximum value of theft: $f_1(8) = 23$

- Metals to take to achieve this maximum value:

$x_1 = 1$, $x_2 = 0$, $x_3 = 1$ $\Longrightarrow$ Take metals 1 and 3

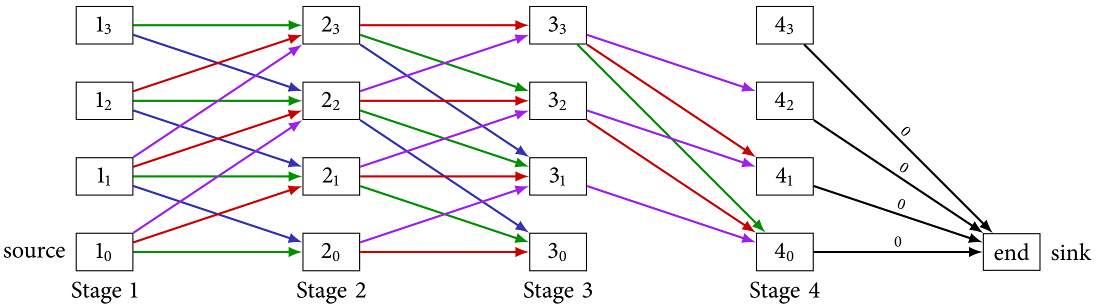
3 Another example

Example 2. The Dijkstra Brewing Company is planning production of its new limited run beer, Primal Pilsner. The company must supply 1 batch next month, then 2 and 4 in successive months. Each month in which the company produces the beer requires a factory setup cost of \$5,000. Each batch of beer costs \$2,000 to produce. Batches can be held in inventory at a cost of \$1,000 per batch per month. Capacity limitations allow a maximum of 3 batches to be produced during each month. In addition, the size of the company's warehouse restricts the ending inventory for each month to at most 3 batches. The company has no initial inventory.

The company wants to find a production plan that will meet all demands on time and minimizes its total production and holding costs over the next 3 months. Formulate this problem as a dynamic program by giving its recursive representation. Solve the dynamic program.

Formulating the DP

- Back in Lesson 5, we formulated this problem as a dynamic program with the following shortest path representation:
  - Stage t represents the beginning of month t ($t = 1, 2, 3$) or the end of the decision-making process ($t = 4$).
  - Node t_n represents having n batches in inventory at stage t ($n = 0, 1, 2, 3$).



Month	Production amount	Edge		Edge length
1	0	$(1_n, 2_{n-1})$	for $n = 1, 2, 3$	$1(n-1)$
1	1	$(1_n, 2_n)$	for $n = 0, 1, 2, 3, 4$	$5 + 2(1) + 1(n)$
1	2	$(1_n, 2_{n+1})$	for $n = 0, 1, 2$	$5 + 2(2) + 1(n+1)$
1	3	$(1_n, 2_{n+2})$	for $n = 0, 1$	$5 + 2(3) + 1(n+2)$
2	0	$(2_n, 3_{n-2})$	for $n = 2, 3$	$1(n-2)$
2	1	$(2_n, 3_{n-1})$	for $n = 1, 2, 3$	$5 + 2(1) + 1(n-1)$
2	2	$(2_n, 3_n)$	for $n = 0, 1, 2, 3$	$5 + 2(2) + 1(n)$
2	3	$(2_n, 3_{n+1})$	for $n = 0, 1, 2$	$5 + 2(3) + 1(n+1)$
3	0	not possible		
3	1	$(3_n, 4_{n-3})$	for $n = 3$	$5 + 2(1) + 1(n-3)$
3	2	$(3_n, 4_{n-2})$	for $n = 2, 3$	$5 + 2(2) + 1(n-2)$
3	3	$(3_n, 4_{n-1})$	for $n = 1, 2, 3$	$5 + 2(3) + 1(n-1)$

- Let d_t = number of batches required in month t, for $t = 1, 2, 3$
- Stages:

$$\text{stage } t \longleftrightarrow \begin{cases} \text{beginning of month } t & t = 1, 2, 3 \\ \text{end of decision-making process} & t = 4 \end{cases}$$

- States:

$$\text{state } n \longleftrightarrow n \text{ batches in inventory} \qquad n = 0, 1, 2, 3$$

- Allowable decisions x_t at stage t and state n:

x_t must satisfy: $x_t \in \{0, 1, 2, 3\}$ ← production capacity

$$0 \underset{\text{satisfy demand}}{\le} \underbrace{n + x_t - d_t}_{\text{inventory at end of month } t} \underset{\text{inventory capacity}}{\le} 3 \qquad \text{for } t = 1, 2, 3 \quad n = 0, 1, 2, 3$$

$n + x_t \geqslant d_t$

- Contribution of decision x_t at stage t and state n:

$$5I(x_t) + 2x_t + 1(n + x_t - d_t) \quad \text{for } t = 1, 2, 3 \quad n = 0, 1, 2, 3$$

where $I(x_t) = \begin{cases} 1 & \text{if } x_t > 0 \\ 0 & \text{o/w} \end{cases}$

- Value-to go function $f_t(n)$ at stage t and state n:

$f_t(n)$ = minimum cost of meeting demand in months $t, \dots, 3$ with initial inventory of n batches, for $t = 1, \dots, 4$, $n = 0, \dots, 3$

- Boundary conditions: $f_4(n) = 0$ for $n = 0, 1, 2, 3$
- Recursion:

$$f_t(n) = \min_{\substack{x_t \in \{0,1,2,3\} \\ 0 \le n + x_t - d_t \le 3}} \left\{ 5I(x_t) + 2x_t + 1(n + x_t - d_t) + f_{t+1}(n + x_t - d_t) \right\}$$

for $t = 1, 2, 3$; $n = 0, 1, 2, 3$

- Desired value-to-go function value: $f_1(0)$

$$f_t(n) = \min_{\substack{x_t \in \{0,1,2,3\} \\ 0 \le n + x_t - d_t \le 3}} \left\{ 5I(x_t) + 2x_t + 1(n + x_t - d_t) + f_{t+1}(n + x_t - d_t) \right\}$$

Solving the DP

- Stage 4 computations – boundary conditions:

$$f_4(n) = 0 \qquad n = 0, 1, 2, 3$$

- Stage 3 computations:

x_3 can be 1, 2, 3

$$f_3(3) = \min_{\substack{x_3 \in \{0,1,2,3\} \\ 0 \le 3 + x_3 - 4 \le 3}} \left\{ 5I(x_3) + 2x_3 + 1(3 + x_3 - 4) + f_4(3 + x_3 - 4) \right\}$$

$$= \min\left\{ 5 + 2(1) \overset{7}{+} 1(0) + f_4(0),\ 5 + 2(2) \overset{10}{+} 1(1) + f_4(1),\ 5 + 2(3) \overset{13}{+} 1(2) + f_4(2) \right\} = 7$$

$0 \le 2 + x_3 - 4 \le 3 \rightarrow$
$\Rightarrow x_3$ can be 2, 3

$$f_3(2) = \min\left\{ 5 + 2(2) \overset{9}{+} 1(0) + f_4(0),\ 5 + 2(3) \overset{12}{+} 1(1) + f_4(1) \right\} = 9$$

$0 \le 1 + x_3 - 4 \le 3 \rightarrow$
$\Rightarrow x_3$ can be 3

$$f_3(1) = \min\left\{ 5 + 2(3) \underset{x_3 = 3}{\overset{11}{+}} 1(0) + f_4(0) \right\} = 11$$

$0 \le 0 + x_3 - 4 \le 3$
$\Rightarrow x_3$ can't be anything

$$f_3(0) = \min\{\ \} = +\infty \leftarrow \text{no allowable decisions}$$

- Stage 2 computations:

$0 \le 3 + x_2 - 2 \le 3 \rightarrow$
$\Rightarrow x_2$ can be 0, 1, 2

$$f_2(3) = \min\left\{ 2(0) \overset{12}{+} 1(1) + f_3(1),\ 5 + 2(1) \overset{18}{+} 1(2) + f_3(2),\ 5 + 2(2) \overset{19}{+} 1(3) + f_3(3) \right\} = 12$$

$0 \le 2 + x_2 - 2 \le 3 \rightarrow$
$\Rightarrow x_2$ can be 0, 1, 2, 3

$$f_2(2) = \min\left\{ 1(0) \overset{+\infty}{+} f_3(0),\ 5 + 2(1) \overset{19}{+} 1(1) + f_3(1),\ 5 + 2(2) \overset{20}{+} 1(2) + f_3(2),\ 5 + 2(3) \overset{21}{+} 1(3) + f_3(3) \right\} = 19$$

$0 \le 1 + x_2 - 2 \le 3 \rightarrow$
$\Rightarrow x_2$ can be 1, 2, 3

$$f_2(1) = \min\left\{ 5 + 2(1) \overset{+\infty}{+} 1(0) + f_3(0),\ 5 + 2(2) \overset{21}{+} 1(1) + f_3(1),\ 5 + 2(3) \overset{22}{+} 1(2) + f_3(2) \right\} = 21$$

$0 \le 0 + x_2 - 2 \le 3 \rightarrow$
$\Rightarrow x_2$ can be 2, 3

$$f_2(0) = \min\left\{ 5 + 2(2) \overset{+\infty}{+} 1(0) + f_3(0),\ 5 + 2(3) \underset{x_2 = 3}{\overset{23}{+}} 1(1) + f_3(1) \right\} = 23$$

- Stage 1 computations – desired value-to-go function:

$0 \le 0 + x_1 - 1 \le 3 \rightarrow$
$\Rightarrow x_1$ can be 1, 2, 3

$$f_1(0) = \min\left\{ 5 + 2(1) \underset{x_1 = 1}{\overset{30}{+}} 1(0) + f_2(0),\ 5 + 2(2) \overset{31}{+} 1(1) + f_2(1),\ 5 + 2(3) \overset{32}{+} 1(2) + f_2(2) \right\}$$

$$= 30$$

- Minimum total production and holding cost: $f_1(0) = 30$

- Production amounts that achieve this minimum value:

$x_1 = 1,\quad x_2 = 3,\quad x_3 = 3 \quad \Rightarrow$ Produce 1 batch in month 1
3 batches in months 2 and 3

A Problems

Problem 1 (Dynamic Distillery – recursion)**.** You have been put in charge of launching Dynamic Distillery's new bourbon whiskey. There are 4 nonoverlapping phases: research, development, manufacturing system design, and initial production and distribution. Each phase can conducted the two speeds: normal or priority. The times required (in months) to complete each phases at the two speeds are:

Level	Research	Development	Manufacturing System Design	Initial Production and Distribution
Normal	4	3	5	2
Priority	2	2	3	1

The costs (in millions of $) of complete each phase at the two speeds are:

Level	Research	Development	Manufacturing System Design	Initial Production and Distribution
Normal	2	2	3	1
Priority	3	3	4	2

You have been given $10 million to execute the launch as quickly as possible. Formulate this problem as a dynamic program by giving its recursive representation. Solve the dynamic program.

Problem 2 (Pear Computers – recursion)**.** Pear Computers has a contract to deliver the following number of laptop computers during the next three months:

	Month 1	Month 2	Month 3
Laptop computers required	200	300	200

For each laptop produced during months 1 and 2, a $100 cost is incurred; for each laptop produced during month 3, a $120 cost is incurred. Each month in which the company produces laptops requires a factory setup cost of $2,500. Laptops can be held in a warehouse at a cost of $15 for each laptop in inventory at the end of a month. The warehouse can hold at most 400 laptops.

Laptops made during a month may be used to meet demand for that month or any future month. Manufacturing constraints require that laptops be produced in multiples of 100, and at most 300 laptops can be produced in any month. The company's goal is to find a production plan that will meet all demands on time and minimizes its total production and holding costs over the next 3 months. Formulate this problem as a dynamic program by giving its recursive representation. Solve the dynamic program.